

# GENERALIZED ITERATED WREATH PRODUCTS OF SYMMETRIC GROUPS AND GENERALIZED ROOTED TREES CORRESPONDENCE

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**ABSTRACT.** Consider the generalized iterated wreath product  $S_{r_1} \wr \dots \wr S_{r_k}$  of symmetric groups. We find a bijection between the equivalence classes of irreducible representations of the generalized iterated wreath product with orbits of labels on certain rooted trees. We find a recursion for the number of these labels and the degrees of irreducible representations of the generalized iterated wreath product. We give upper bound estimates for fast Fourier transforms.

## 1. INTRODUCTION

Wreath products of symmetric groups arise as the automorphism groups of regular rooted trees, with applications ranging from functions on rooted trees, pixel blurring, nonrigid molecules, to choosing subcommittees from a set of committees. With motivation from [OOR04], we find a bijection between the equivalence classes of irreducible representations of the generalized iterated wreath product  $W(\vec{r}|_k)$  and the orbits of families of labels on certain complete trees. We find a recursion for the number of equivalence classes and dimensions of the irreducible representations of the generalized iterated wreath products. We also give upper bound estimates for fast Fourier transforms of this chain of groups.

**1.1. Background and Notation.** Throughout this paper, we will fix  $\vec{r} = (r_1, r_2, r_3, \dots) \in \mathbb{N}^\omega$ , a positive integral vector. We denote by  $\vec{r}|_k := (r_1, r_2, \dots, r_k)$  the  $k$ -length vector found by truncating  $\vec{r}$ . For a group  $G$ , we denote by  $\widehat{G}$  the set of irreducible representations of  $G$ . We say that  $\mathcal{R}_G$  is a *traversal* for  $G$  if  $\mathcal{R}_G \subset \widehat{G}$  contains exactly one irreducible for each isomorphism class. A consequence of basic representation theory is that  $\sum_{\rho \in \mathcal{R}} \dim(\rho)^2 = |G|$ .

**Definition 1.1.** Let the (generalized)  $k$ -th  $\vec{r}$ -symmetric wreath product  $W(\vec{r}|_k)$  be defined recursively by

$$W(\vec{r}|_0) = 1 \text{ and } W(\vec{r}|_k) = W(\vec{r}|_{k-1}) \wr S_{r_k}.$$

Note that  $W(\vec{r}|_1) = S_{r_1}$ ,  $W(\vec{r}|_2) = S_{r_1} \wr S_{r_2}$ , and  $W(\vec{r}|_k) = S_{r_1} \wr \dots \wr S_{r_k}$ . Throughout this paper, we will be considering the chain of groups given in Definition 1.1.

## 2. IRREDUCIBLE REPRESENTATIONS OF ITERATED WREATH PRODUCTS

Let  $\text{Ind}_H^K \sigma$  denote the representation of  $K$  with dimension  $[K : H] \cdot \dim \sigma$  induced from representation  $\sigma \in \widehat{H}$  of subgroup  $H \leq K$ . Let  $\alpha \vdash_h n$  denote that  $\alpha = (\alpha_1, \dots, \alpha_h) \in (\mathbb{Z}_{\geq 0})^h$  is a weak composition of  $n$  with  $h$  parts so that  $\alpha$  satisfies  $\sum_{i=1}^h \alpha_i = n$ .

**Theorem 2.1.** If  $\mathcal{R} = \{\rho_1, \dots, \rho_h\}$  is a traversal for  $G \leq S_N$ , then the irreducible representations given by

$$\left\{ \text{Ind}_{G \wr S_\alpha}^{G \wr S_n} (\rho_1^{\alpha_1} \otimes \dots \otimes \rho_h^{\alpha_h} \otimes \sigma) \mid \alpha \vdash_h n, \sigma \in \widehat{S_\alpha} \right\}$$

form a traversal of  $G \wr S_n$ , where  $S_\alpha := S_{\alpha_1} \times S_{\alpha_2} \times \dots \times S_{\alpha_h}$  with  $S_0 = 1$ . In particular, if  $\mathcal{R} = \{\rho_1, \dots, \rho_h\}$  is a traversal for  $W(\vec{r}|_{k-1})$ , then the following set forms a traversal of  $W(\vec{r}|_k)$ :

$$\mathcal{R}_{W(\vec{r}|_k)} := \left\{ \text{Ind}_{W(\vec{r}|_{k-1}) \wr S_\alpha}^{W(\vec{r}|_k)} (\rho_1^{\alpha_1} \otimes \dots \otimes \rho_h^{\alpha_h} \otimes \sigma) \mid \alpha \vdash_h n, \sigma \in \widehat{S_\alpha} \right\}. \quad (1)$$

*Proof.* The set  $\{\rho^\alpha = \rho_1^{\alpha_1} \otimes \dots \otimes \rho_h^{\alpha_h} : \alpha \vdash_h n\}$  forms a traversal for  $G^n$ . The theorem follows as a consequence of Clifford theory to the structure of irreducible representations of  $G \wr S_n = G^n \rtimes S_n$ .  $\square$

Let  $N(\vec{r}|_k)$  denote the number of irreducible representations of  $W(\vec{r}|_k)$ .

**Corollary 2.2.** Define  $P(n, h) := \sum_{\alpha \vdash_h n} \prod_{i=1}^h \alpha_i!$ . We find that  $N(\vec{r}|_k)$  satisfies the following recursion:

$$N(\vec{r}|_k) = P(r_k, N(\vec{r}|_{k-1})) = \sum_{k=0}^h k! \cdot P(n-k, h-1). \quad (2)$$

### 3. BRANCHING DIAGRAM TO $\vec{r}$ -LABEL CORRESPONDENCE

We find a combinatorial structure describing the branching diagrams for the iterated wreath products of symmetric groups.

**Definition 3.1.** We define the complete  $\vec{r}$ -tree  $T(\vec{r}|_k)$  of height  $k$ , or  $\vec{r}|_k$ -tree, recursively as follows. Let  $T(r_1)$  be the tree consisting of a root node only. Let  $T(\vec{r}|_k)$  consist of a root node with  $r_k$  children, with each the root of a copy of the  $k-1$ -level tree  $T(\vec{r}|_{k-1})$ , which yields a tree with  $k$  levels of nodes.

Notice that  $T(\vec{r}|_k)$  has  $\prod_{i=2}^k r_i$  leaves. We say a node  $v$  is in the  $j$ -th layer of  $T(\vec{r}|_k)$  if it is at distance  $j$

from the root. The tree  $T(\vec{r}|_k)$  has  $\prod_{i=k-j+1}^k r_i$  nodes in the  $j$ -th layer. The subtree of  $T$  rooted at some vertex  $v$  denoted  $T_v$  is the tree with root  $v$  consisting of all the children and descendants of  $v$ . We call  $T_v$  a maximal subtree of  $T$  if  $v$  is a child of the root, or equivalently if  $v$  is in the first layer. Let  $\deg(v)$  denote the number of leaves of the subtree  $T_v$ , and let  $[n] := \{1, 2, \dots, n\}$  be the set of integers from 1 to  $n$ .

We denote by  $\hat{S}_* := \bigsqcup_{n \in \mathbb{N}} \bigsqcup_{\alpha \models n} \hat{S}_\alpha$ , where  $\models$  denotes “is a partition of”.

**Definition 3.2.** An  $\vec{r}|_k$ -label is a function  $\phi : V_{T(\vec{r}|_k)} \rightarrow \hat{S}_*$  satisfying  $\phi(v) \in \bigsqcup_{\alpha \vdash \deg(v)} \hat{S}_\alpha$ .

We say that two labels  $\phi$  and  $\psi$  on  $T$  are equivalent, or  $\phi \sim \psi$ , if there exists  $\sigma \in \text{Aut}(T)$  such that  $\phi^\sigma = \psi$ , where  $\phi^\sigma(v) := \phi(v^\sigma)$ .

**Definition 3.3.** An  $\vec{r}|_k$ -label  $\phi : V_{T(\vec{r}|_k)} \rightarrow \hat{S}_*$  is valid if it satisfies all of the following recursive conditions. We denote by  $\mathcal{T}(\vec{r}|_k) := \{\phi : \phi \text{ is a valid } \vec{r}|_k\text{-label}\}$  and  $\mathcal{T} = \bigsqcup_k \mathcal{T}(\vec{r}|_k)$ .

- Given an  $\vec{r}|_1$ -label  $\phi : V_{T(\vec{r}|_1)} = \{\text{root node}\} \rightarrow \hat{S}_*$ : we require  $\phi \in \hat{S}_{r_1}$ .
- Given for  $k > 1$  an  $\vec{r}|_k$ -label  $\phi : V_{T(\vec{r}|_k)} \rightarrow \hat{S}_*$ : we require
  - (1) for any child  $v$  of the root, the  $\vec{r}|_{k-1}$ -label  $\phi|_{T_v} \in \mathcal{T}_{k-1}$ , and
  - (2)  $\phi(\text{root node}) \in \hat{S}_\alpha$ , where  $S_\alpha$  gives the stabilizer of the action by  $S_{r_k}$  on  $\vec{r}|_{k-1}$  sublabels of  $\phi$ , so that  $\alpha \models r_k$  is the partition of  $[r_k]$  given by the number of  $\vec{r}|_{k-1}$ -sublabels of  $\phi$  in each non-empty equivalence class,
 where  $\phi|_{T_v}$  denotes the restriction of  $\phi$  to the subtree  $T_v$ .

**Theorem 3.4.** There is a bijection between equivalence classes of  $\widehat{W}(\vec{r}|_k)$  and  $W(\vec{r}|_k)$ -orbits of  $\mathcal{T}(\vec{r}|_k)$ .

*Proof.* It suffices to define a map only on a traversal of  $\widehat{W}(\vec{r}|_k)$  such as that given in (1). We will define  $F : \mathcal{R}_{W(\vec{r}|_k)} \rightarrow \mathcal{T}(\vec{r}|_k)$  recursively and it suffices to prove that each orbit of  $\mathcal{T}(\vec{r}|_k)$  under action by  $W(\vec{r}|_k)$  has exactly one pre-image under  $F$ .

Let  $k = 1$ . For any  $\rho \in \widehat{W}(\vec{r}|_1) = \hat{S}_{r_1}$ , we define the  $\vec{r}|_1$ -label  $F(\rho) : V_{T(\vec{r}|_1)} = \{\text{root}\} \rightarrow \hat{S}_{r_1}$  as  $F(\rho)(\text{root}) := \rho$ . This is clearly a bijection as desired.

Now let  $k > 1$ . By the inductive hypothesis,  $F : \mathcal{R}_{W(\vec{r}|_{k-1})} \rightarrow \mathcal{T}(\vec{r}|_{k-1})$  has exactly one pre-image per orbit of  $\mathcal{T}(\vec{r}|_{k-1})$ . Suppose that a traversal for  $W(\vec{r}|_{k-1})$  is given by the set  $\{\rho_1, \dots, \rho_h\}$ . We need to define  $F$  on  $\mathcal{R}_{W(\vec{r}|_k)}$  and show that orbits have exactly one pre-image as desired.

Pick an arbitrary element of  $\rho_1^{\alpha_1} \otimes \dots \otimes \rho_h^{\alpha_h} \otimes \sigma$  of  $\mathcal{R}_{W(\vec{r}|_k)}$ . Denote its image under  $F$  by  $\phi := F(\rho_1^{\alpha_1} \otimes \dots \otimes \rho_h^{\alpha_h} \otimes \sigma) : V_{T(\vec{r}|_k)} \rightarrow S_{r_k}$ . Let  $U \subset V_{T(\vec{r}|_k)}$  be the  $r_k$  children of the root. Assign

an ordering to  $U = \{u_1, \dots, u_{r_k}\}$ ; then partition  $U$  as  $U = U_1 \sqcup \dots \sqcup U_h$  satisfying  $|U_i| = \alpha_i$  while preserving the ordering. For each  $u_i \in U$ , define the value of  $\phi$  on all nodes in subtree  $T_{u_i}$  to satisfy  $\phi|_{u_i} = F(\rho_{j^i})$ , where  $j^i$  satisfies  $U_{j^i} \ni u_i$  and where  $\phi|_{T_{u_i}}$  denotes the restriction of  $\phi$  to  $T_{u_i} \subseteq T$ . It remains to define the value of  $\phi$  on the root node. We let  $\phi(\text{root}) = \sigma$ .

Notice that  $\phi_{u_i} \in \mathcal{T}(\vec{r}|_{k-1})$  by definition and induction. Since  $\sigma$  is in the stabilizer of the action by  $S_{r_k}$  on  $\rho^\alpha$ , which is exactly  $S_\alpha$ ,  $\phi$  is a compatible label for  $T(\vec{r}|_k)$ . Thus,  $F$  is well-defined and clearly each orbit of  $\mathcal{T}(\vec{r}|_k)$  has exactly one pre-image.  $\square$

**3.1. Degrees of Irreducible Representations.** Following the discussion in [OOR04], we define for any  $\vec{r}|_k$ -tree  $T$  the companion tree  $C_T$ .

**Definition 3.5.** Fix  $T(\vec{r}|_k)$  and  $\vec{r}|_k$ -label  $\phi$ . Let the companion label  $C_\phi : V_{T(\vec{r}|_k)} \rightarrow \mathbb{N}$  be defined by:

$$C_\phi(v) = \begin{cases} \dim(\phi(v)) & \text{if } v \text{ is a leaf of } T(\vec{r}|_k), \\ |S_{r_i}/S_\alpha| = \binom{r_i}{\alpha} & \text{otherwise, where } v \text{ is in the } (k-i)\text{-th layer of } T \text{ and } \phi(v) \in S_\alpha. \end{cases}$$

**Proposition 3.6.** If  $\rho$  is an irreducible representation of  $W(\vec{r}|_k)$  associated to  $\vec{r}|_k$ -label  $\phi$ , then the dimension  $d_\rho$  of  $\rho$  is given by

$$d_\rho = \prod_v C_\phi(v). \quad (3)$$

the product of the value of the companion label  $C_\phi$  on all vertices.

#### 4. FAST FOURIER TRANSFORMS, ADAPTED BASES AND UPPER BOUND ESTIMATES

We use the FFT estimates derived in [CB93] and [Roc95] to find an overall upper bound on the running time of finding an FFT for  $W(\vec{r}|_k)$ . From [Roc95], we cite the result:

**Theorem 4.1.** We have

$$T(G \wr S_n) \leq nT(G) \cdot |G \wr S_{n-1}| + nT(G \wr S_{n-1}) \cdot |G| + n^3 2^{|\hat{G}|} |G \wr S_n|. \quad (4)$$

**4.1. Acknowledgment.** The authors would like to acknowledge Mathematics Research Communities for providing the authors with an exceptional working environment at Snowbird, Utah and they would like to thank Michael Orrison for helpful discussions. This paper was written during the first author's visit to the University of Chicago in 2014. She thanks their hospitality.

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